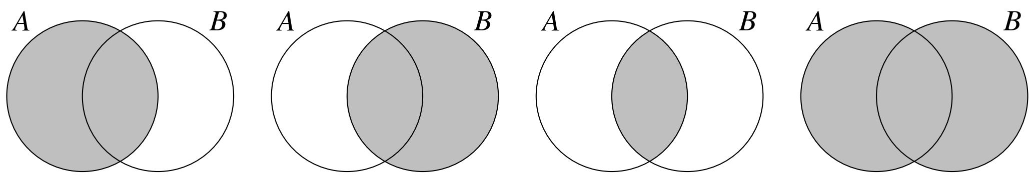
Counting Intro II

Note 10

Inclusion-exclusion: With two sets,



$|A| \quad + \quad |B| \quad - \quad |A \cap B| \quad = \quad |A \cup B|$

With more sets,

$$\begin{aligned}|A_1 \cup A_2 \cup \cdots \cup A_n| = &|A_1| + |A_2| + \cdots + |A_n| \\ &- |A_1 \cap A_2| - |A_1 \cap A_3| - \cdots - |A_i \cap A_j| - \cdots - |A_{n-1} \cap A_n| \\ &+ |A_1 \cap A_2 \cap A_3| + \cdots + |A_i \cap A_j \cap A_k| + \cdots + |A_{n-2} \cap A_{n-1} \cap A_n| \\ &\cdots\end{aligned}$$

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{k=1}^{n} (-1)^{k-1} \sum_{\substack{S \subseteq \{1,\ldots,n\} \\ |S|=k}} \left|\bigcap_{i \in S} A_i\right|$$

That is, for each size k, iterate through all ways of picking k sets from $\{A_1, \ldots, A_n\}$, and alternate between adding and subtracting the sizes of their intersection.

Combinatorial proofs: A technique for proving combinatorial identities. There should be very little math involved (usually none): use two different ways of counting the same scenario. One way should correspond to the left-hand side of the equality, and the other way should correspond to the right-hand side of the equality. The fact that we're counting the same scenario means that the two sides are equal.

1 Inclusion and Exclusion

Note 10

What is the total number of positive integers strictly less than 60 that are also coprime to 60?

2 Fibonacci Fashion

Note 10

You have n accessories in your wardrobe, and you'd like to plan which ones to wear each day for the next t days. As a student of the Elegant Etiquette Charm School, you know it isn't fashionable to wear the same accessories multiple days in a row. (Note that the same goes for clothing items in general). Therefore, you'd like to plan which accessories to wear each day represented by subsets $S_1, S_2, \ldots, S_t$, where $S_1 \subseteq \{1, 2, \ldots, n\}$ and for $2 \le i \le t$, $S_i \subseteq \{1, 2, \ldots, n\}$ and S_i is disjoint from S_{i-1}.

(a) For $t \ge 1$, prove that there are F_{t+2} binary strings of length t with no consecutive zeros (assume the Fibonacci sequence starts with $F_0 = 0$ and $F_1 = 1$).

(b) Use a combinatorial proof to prove the following identity, which, for $t \ge 1$ and $n \ge 0$, gives the number of ways you can create subsets of your n accessories for the next t days such that no accessory is worn two days in a row:

$$\sum_{x_1 \ge 0} \sum_{x_2 \ge 0} \cdots \sum_{x_t \ge 0} \binom{n}{x_1} \binom{n - x_1}{x_2} \binom{n - x_2}{x_3} \cdots \binom{n - x_{t-1}}{x_t} = (F_{t+2})^n.$$

(You may assume that $\binom{a}{b} = 0$ whenever $a < b$.)

3 CS70: The Musical

Note 10

Edward, one of the previous head TA's, has been hard at work on his latest project, *CS70: The Musical*. It's now time for him to select a cast, crew, and directing team to help him make his dream a reality.

(a) First, Edward would like to select directors for his musical. He has received applications from $2n$ directors. Use this to provide a combinatorial argument that proves the following identity:

$$\binom{2n}{2} = 2\binom{n}{2} + n^2.$$

(b) Edward would now like to select a crew out of n people. Use this to provide a combinatorial argument that proves the following identity: (this is called Pascal's Identity)

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

(c) There are n actors lined up outside of Edward's office, and they would like a role in the musical (including a lead role). However, he is unsure of how many individuals he would like to cast. Use this to provide a combinatorial argument that proves the following identity:

$$\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$$

(d) Generalizing the previous part, provide a combinatorial argument that proves the following identity:

$$\sum_{k=j}^{n} \binom{n}{k}\binom{k}{j} = 2^{n-j}\binom{n}{j}.$$

4 August Absurdity

Since March Madness was cancelled, the council unanimously decided to have August Absurdity instead - an online Discrete Mathematics tournament! There are 64 teams (including Cal) in the single-elimination tournament - that means, every match is between two teams and will decide a winner who moves on to the next round and a loser who is eliminated from the tournament. Thus the first round will have 64 teams, the next will have 32, and so on until 1 remains. There is a single, randomly initialized, starting bracket.

(a) How many tournament outcomes exist such that Cal wins the entire tournament?

(b) In the first round, Cal will face a no-name school called LJSU (some people call it Stanfurd?). The format of each match is as follows: Each of the two teams have 8 players labelled from 1 to 8. They play a series of games. In the first game, the two 1's play each other. The loser of the game is eliminated and replaced by the next player of the same team until all players from one team are eliminated, ending the match. What is the number of possible sequences of games such that Cal wins the match?

(c) Cal employs a blasphemous strategy that even baffles themselves. They place their players in an order such that each player is either taller than all the preceding players or shorter than all the preceding players. Let 1-8 represent the players' heights. An example of a valid ordering: 4, 5, 6, 3, 2, 7, 1, 8. An example of an invalid ordering: 1, 2, 3, 4, 5, 6, 8, 7. (invalid since 7 is neither taller or shorter than all the preceding players). How many such orderings exist?

(d) To keep viewership up after the tournament finishes, the council plans an All-Star match. The 16 greatest players in the league were chosen, including Oski and a tree..? Oski refuses to play on the same team as the tree. How many ways can the 16 players be distributed into two teams of 8 players such that Oski and the tree are in opposite teams?

(e) Provide an explanation for the following combinatorial identity. Hint: Solve the previous part using another method. Those two methods should correspond to the two sides of the equality.

$$\binom{n}{r} - \binom{n-2}{r-2} - \binom{n-2}{r} = 2\binom{n-2}{r-1}$$