Discrete Probability Intro

Note 13

Probability Space: A probability space is a tuple $(\Omega, \mathbb{P})$, where Ω is the *sample space* and $\mathbb{P}$ is the *probability function* on the sample space.

Specifically, Ω is the set of all outcomes ω, and $\mathbb{P}$ is a function $\mathbb{P}\colon \Omega \to [0,1]$, assigning a probability to each outcome, satisfying the following conditions:

$$0 \le \mathbb{P}[\omega] \le 1 \quad \text{and} \quad \sum_{\omega \in \Omega} \mathbb{P}[\omega] = 1.$$

Event: an event A is a subset of Ω, i.e. a collection of some outcomes in the sample space. We define

$$\mathbb{P}[A] = \sum_{\omega \in A} \mathbb{P}[\omega].$$

Uniform Probability Space: all outcomes are assigned the same probability, i.e. $\mathbb{P}[\omega] = \frac{1}{|\Omega|}$; this is just counting!

With an event A in a uniform probability space, $\mathbb{P}[A] = \frac{|A|}{|\Omega|}$, which is again more counting!

1 Symmetry

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In this problem, we will walk you through the idea of *symmetry* and its formal justification. Consider an experiment where you have a bag with m red marbles and $n-m$ blue marbles. You draw marbles from the bag, one at a time without replacement until the bag is empty.

(a) Define the sample space Ω. (No need to write out every element, a brief description is fine). Is this a uniform probability space?

(b) What is the probability that the first marble you draw is red?

(c) Suppose you've drawn all but the final marble, setting each marble aside as you draw it *without looking at it*. We want to find the probability that the final marble left in the bag will be red.

Let A be the event containing outcomes where the first marble is red, and let B be the event containing outcomes where the final marble is red. Provide a bijective function $f : A \to B$ mapping outcomes in A to outcomes in B, and explain why it is a bijection. Note that there can be multiple valid bijections.

(d) Use the previous parts to find the probability that the final marble will be red.

(e) You repeat the experiment. Find the probability that the last two marbles you draw will be red.

(f) You repeat the experiment again, but this time you see that the first marble you draw is red. Find the probability that the second-to-last marble you draw will also be red.

2 Flippin' Coins

Note 13

Suppose we have an unbiased coin, with outcomes H and T, with probability of heads $\mathbb{P}[H] = 1/2$ and probability of tails also $\mathbb{P}[T] = 1/2$. Suppose we perform an experiment in which we toss the coin 3 times. An outcome of this experiment is (X_1, X_2, X_3), where $X_i \in \{H, T\}$.

(a) What is the *sample space* for our experiment?

(b) Which of the following are examples of *events*? Select all that apply.

- $\{(H,H,T),(H,H),(T)\}$
- $\{(T,H,H),(H,T,H),(H,H,T),(H,H,H)\}$
- $\{(T,T,T)\}$
- $\{(T,T,T),(H,H,H)\}$
- $\{(T,H,T),(H,H,T)\}$

(c) What is the complement of the event $\{(H,H,H),(H,H,T),(H,T,H),(H,T,T),(T,T,T)\}$?

(d) Let A be the event that our outcome has 0 heads. Let B be the event that our outcome has exactly 2 heads. What is $A \cup B$?

(e) What is the probability of the outcome (H,H,T)?

(f) What is the probability of the event that our outcome has exactly two heads?

(g) What is the probability of the event that our outcome has at least one head?

3 Sampling

Note 13

Suppose you have balls numbered $1,\ldots,n$, where n is a positive integer ≥ 2, inside a coffee mug. You pick a ball uniformly at random, look at the number on the ball, replace the ball back into the coffee mug, and pick another ball uniformly at random.

(a) What is the probability that the first ball is 1 and the second ball is 2?

(b) What is the probability that the second ball's number is strictly less than the first ball's number?

(c) What is the probability that the second ball's number is exactly one greater than the first ball's number?

(d) Now, assume that after you looked at the first ball, you did *not* replace the ball in the coffee mug (instead, you threw the ball away), and then you drew a second ball as before. Now, what are the answers to the previous parts?

4 Intransitive Dice

Note 13

You're playing a game with your friend Bob, who has a set of three dice. You'll each choose a different die, roll it, and whoever had the higher result wins. The dice have sides as follows:

- Die A has sides 2, 2, 4, 4, 9, and 9.
- Die B has sides 1, 1, 6, 6, 8, and 8.
- Die C has sides 3, 3, 5, 5, 7, and 7.

(a) Suppose you have chosen die A and Bob has chosen die B. What is the probability that you win?
Hint: It may be easier to work with a sample space smaller than 6×6.

(b) Suppose you have chosen die B and Bob has chosen die C. What is the probability that you win?

(c) Suppose you have chosen die C and Bob has chosen die A. What is the probability that you win?

(d) Bob offers to let you choose your die first so that you can choose the best one. Is this an offer your should accept? Why or why not?