Due: Saturday, 7/19, 4:00 PM
Grace period until Saturday, 7/19, 6:00 PM
Remember to show your work for all problems!

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.) If you used an LLM, place transcripts of your chats here.

1 Jelly Bean Factory

Note 10 A candy factory has an endless supply of red, orange and yellow jelly beans. The factory packages the jelly beans into jars of 100 jelly beans each, with each possible combination of colors in the jar being equally likely. (One possible color combination, for example, is a jar of 56 red, 22 orange, and 22 yellow jelly beans.)

Find N, the number of different possible color combinations of jelly beans in a single jar (note that color combinations are unordered).

2 Grids and Trees!

Note 10 Suppose we are given an $n \times n$ grid, for $n \geq 1$, where one starts at $(0,0)$ and goes to (n,n). On this grid, we are only allowed to move left, right, up, or down by increments of 1.

(a) How many shortest paths are there that go from $(0,0)$ to (n,n)?

(b) How many shortest paths are there that go from $(0,0)$ to $(n-1,n+1)$?

Now, consider shortest paths that meet the conditions where we can only visit points (x,y) where $y \leq x$. That is, the path cannot cross line $y = x$. We call these paths n-legal paths for a maze of side length n. Let F_n be the number of n-legal paths.

(c) Compute the number of shortest paths from $(0,0)$ to (n,n) that cross $y = x$. (Hint: Let (i,i) be the first time the shortest path crosses the line $y = x$. Then the remaining path starts from $(i,i+1)$ and continues to (n,n). If in the remainder of the path one exchanges y-direction moves with x-direction moves and vice versa, where does one end up?)

(d) Compute the number of shortest paths from $(0,0)$ to (n,n) that do not cross $y = x$. (You may find your answers from parts (a) and (c) useful.)

(e) A different idea is to derive a recursive formula for the number of paths. Fix some i with $0 \le i \le n-1$. We wish to count the number of n-legal paths where the last time the path touches the line $y = x$ is the point (i,i). Show that the number of such paths is $F_i \cdot F_{n-i-1}$. (Hint: If $i = 0$, what are your first and last moves, and where is the remainder of the path allowed to go?)

(f) Explain why $F_n = \sum_{i=0}^{n-1} F_i \cdot F_{n-i-1}$.

(g) Create and explain a recursive formula for the number of trees with n vertices ($n \ge 1$), where each non-root node has degree at most 3, and the root node has degree at most 2. Two trees are different if and only if either left-subtree is different or right-subtree is different.

(Notice something about your formula and the grid problem. Neat!)

3 Is This CS 61C?

Note 10

XOR ($\oplus$) is a function that takes in two integers that are each either 0 or 1 (also known as **one-bit integers**), and returns 0 if they have the same value, and 1 otherwise. For example, $1 \oplus 0 = 1$, whereas $0 \oplus 0 = 0$. Note that we can generalize XOR for k one-bit integers to say $x_1 \oplus \cdots \oplus x_k = (x_1 + \cdots + x_k) \pmod 2$.

(a) Show that the number of one-bit integer solutions $(x_1, \dots, x_k)$ for $x_1 \oplus \cdots \oplus x_k = 0$ is 2^{k-1}.

We can extend XOR to an n-bit string, which is a length n string where each integer in the string is an one-bit integer. If we have two n-bit strings $s = s_1 \dots s_n$ and $t = t_1 \dots t_n$, we define $s \oplus t$ to be $u_1 \dots u_n$, where for all i, $u_i = s_i \oplus t_i$. We can similarly define XOR for k n-bit strings $y_1 = z_{1,1} \dots z_{1,n}$, $\dots$, $y_k = z_{k,1} \dots z_{k,n}$, and $y_1 \oplus \cdots \oplus y_k = w_1 \dots w_n$, where for all i, $w_i = z_{1,i} \oplus \cdots \oplus z_{k,i}$.

(b) Show that for an arbitrary n-bit string x, the number of n-bit string solutions $(y_1, \dots, y_k)$ for $y_1 \oplus \cdots \oplus y_k = x$ is $2^{(k-1)n}$. (Hint: You may find your answer from part (a) useful.)

4 Code Reachability

Note 12

Consider triplets (M, x, L) where

- `M` is a Java program
- `x` is some input
- `L` is an integer

and the question of: if we execute $M(x)$, do we ever hit line L?

Prove this problem is undecidable.

5 Computations on Programs

Note 12

(a) Is it possible to write a program that takes a natural number n as input, and finds the shortest arithmetic formula which computes n? For the purpose of this question, a formula is a sequence consisting of some valid combination of (decimal) digits, standard binary operators ($+$, $\times$, the "^" operator that raises to a power), and parentheses. We define the length of a formula as the number of characters in the formula. Specifically, each operator, decimal digit, or parentheses counts as one character.

(*Hint:* Think about whether it's possible to enumerate the set of possible arithmetic formulas. How would you know when to stop?)

(b) Now say you wish to write a program that, given a natural number input n, finds another program (e.g. in Java or C) which prints out n. The discovered program should have the minimum execution-time-plus-length of all the programs that print n. Execution time is measured by the number of CPU instructions executed, while "length" is the number of characters in the source code. Can this be done?

(*Hint:* Is it possible to tell whether a program halts on a given input within t steps? What can you say about the execution-time-plus-length of the program if you know that it does not halt within t steps?)

6 Countability: True or False

Note 11

(a) The set of all irrational numbers $\mathbb{R}\backslash\mathbb{Q}$ (i.e. real numbers that are not rational) is uncountable.

(b) The set of real solutions for the equation $x+y=1$ is countable.

For any two functions $f : Y \to Z$ and $g : X \to Y$, let their composition $f \circ g : X \to Z$ be given by $f \circ g = f(g(x))$ for all $x \in X$. Determine if the following statements are true or false.

(c) f and g are injective (one-to-one) $\implies$ $f \circ g$ is injective (one-to-one).

(d) f is surjective (onto) $\implies$ $f \circ g$ is surjective (onto).

7 Counting Shapes

Note 11

Suppose scaled and shifted copies of a shape S are embedded into the plane $\mathbb{R}^2$. Let $\mathscr{C}$ denote the collection of all these copies. Thus each element in $\mathscr{C}$ determines the scaling and the position of that copy. Suppose further that the embedding is such that no two copies intersect. For example in the case of filled squares, if there is any overlap between two squares, then they intersect. In the case of the (non-filled) square, two copies intersect if and only if their boundaries intersect. Similarly, in the case of the halved-square, if either the boundary or the middle line of one square intersects with either the boundary or the middle line of some other square, then these two squares intersect.

Can $\mathscr{C}$ be uncountable if S is

(a) the filled square:  ?

(b) the square: $\square$?

(c) the halved square: $\boxminus$?

If no uncountable $\mathscr{C}$ exists, prove that all $\mathscr{C}$ must be countable.