CS 70 Discrete Mathematics and Probability Theory
Summer 2025 Tate **HW 04**

Due: Saturday, 07/26, 4:00 PM
Grace period until Saturday, 07/26, 6:00 PM
Remember to show your work for all problems!

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.) If you used an LLM, place transcripts of your chats here.

1 Is This EECS 126?

Note 13
Note 14

Youngmin loves birdwatching. There are three sites he birdwatches at: Site A, B, and C. He goes to Site A 60% of the time, Site B 35% of the time, and Site C 5% of the time. The probability of seeing a nightingale at each site is $\frac{1}{10}$, $\frac{3}{10}$, and $\frac{2}{5}$, respectively. Using the information above, answer the following questions.

(a) What is the total probability that Youngmin sees a nightingale?

(b) Given that Youngmin sees a nightingale, what is the probability that Youngmin went to site B?

(c) Given that Youngmin didn't go to site B, what is the probability that Youngmin doesn't see a nightingale?

2 Presidential Election

Note 13

We traveled back in time to 1960 and want to determine which presidential candidate will win this coming election. There are two candidates, John F. Kennedy and Richard Nixon.

1. For each state, Kennedy has probability p of winning. What is the probability that Kennedy will win exactly half the states (there are 50 states)?
2. What is the probability that Kennedy will win at least one state?
3. Say that $p = 0.25$ or $p = 0.75$ with equal chance. If Kennedy wins every single state, what is the probability that $p = 0.75$?

3 Solve the Rainbow

Note 14

Your roommate was having Skittles for lunch and they offer you some. There are five different colors in a bag of Skittles: red, orange, yellow, green, and purple, and there are 20 of each color. You know your roommate is a huge fan of the green Skittles. With probability $1/2$ they ate all of the green ones, with probability $1/4$ they ate half of them, and with probability $1/4$ they only ate 5 green ones.

(a) If you take a Skittle from the bag, what is the probability that it is green?

(b) If you take two Skittles from the bag, what is the probability that at least one is green?

(c) If you take three Skittles from the bag, what is the probability that they are all green?

(d) If all three Skittles you took from the bag are green, what are the probabilities that your roommate had all of the green ones, half of the green ones, or only 5 green ones?

(e) If you take three Skittles from the bag, what is the probability that they are all the same color?

4 Cliques in Random Graphs

Note 13
Note 14

Consider the graph $G = (V,E)$ on n vertices which is generated by the following random process: for each pair of vertices u and v, we flip a fair coin and place an (undirected) edge between u and v if and only if the coin comes up heads.

(a) What is the size of the sample space?

(b) A k-clique in a graph is a set S of k vertices which are pairwise adjacent (every pair of vertices is connected by an edge). For example, a 3-clique is a triangle. Let E_S be the event that a set S forms a clique. What is the probability of E_S for a particular set S of k vertices?

(c) Suppose that $V_1 = \{v_1, \ldots, v_\ell\}$ and $V_2 = \{w_1, \ldots, w_k\}$ are two arbitrary sets of vertices. What conditions must V_1 and V_2 satisfy in order for E_{V_1} and E_{V_2} to be independent? Prove your answer.

(d) Prove that $\binom{n}{k} \leq n^k$. (You might find this useful in part (e)).

(e) Prove that the probability that the graph contains a k-clique, for $k \geq 4\log_2 n + 1$, is at most $1/n$. *Hint:* Use the union bound.

5 Pairwise Independence

Note 14

The events A_1, A_2, A_3 are *pairwise independent* if, for all $i \neq j$, A_i is independent of A_j. However, pairwise independence is a weaker statement than *mutual independence*, which requires the additional condition that $\mathbb{P}(A_1, A_2, A_3) = \mathbb{P}(A_1)\,\mathbb{P}(A_2)\,\mathbb{P}(A_3)$.

Try to construct an example where three events are pairwise independent but not mutually independent.

Here is one potential starting point: Let A_1, A_2 be the respective results of flipping two fair coins. Can you come up with an event A_3 that works?

6 Independent Complements

Note 14

Let Ω be a sample space, and let $A, B \subseteq \Omega$ be two independent events.

(a) Prove or disprove: $\overline{A}$ and $\overline{B}$ must be independent.

(b) Prove or disprove: A and $\overline{B}$ must be independent.

(c) Prove or disprove: A and $\overline{A}$ must be independent.

(d) Prove or disprove: It is possible that $A = B$.

7 Pairs of Beads

Note 14

Sinho has a set of $2n$ beads ($n \geq 2$) of n different colors, such that there are two beads of each color. He wants to give out pairs of beads as gifts to all the other $n-1$ TAs, and plans on keeping the final pair for himself (since he is, after all, also a TA). To do so, he first chooses two beads at random to give to the first TA he sees. Then he chooses two beads at random from those remaining to give to the second TA he sees. He continues giving each TA he sees two beads chosen at random from his remaining beads until he has seen all $n-1$ TAs, leaving him with just the two beads he plans to keep for himself. Prove that the probability that at least one of the other TAs (*not* including Sinho himself) gets two beads of the same color is at most $\frac{1}{2}$.