Error Correcting Codes

CS70: Discrete Mathematics and Probability Theory

UC Berkeley – Summer 2025

Lecture 11

Ref: Note 9

Today

Last time:

Shared (and sort of kept) secrets

Today: Dealing with errors

Tolerate (identified) loss: erasure codes

Tolerate (unidentified) corruption: error correcting codes

... using a beautiful decoding algorithm

Review: Interpolation via Linear Equations

Problem: Find coefficients for $P(x) = a_{k-1}x^{k-1} + a_{k-2}x^{k-2} + \cdots + a_1 x + a_0$ going through points $(x_1, y_1); (x_2, y_2) \cdots (x_k, y_k)$.

... k points gives degree (at most) $k-1$ polynomial – working mod p:

$$
\begin{aligned}
a_{k-1}x_1^{k-1} + \cdots + a_0 &\equiv y_1 \pmod{p} \\
a_{k-1}x_2^{k-1} + \cdots + a_0 &\equiv y_2 \pmod{p} \\
\vdots \qquad &\vdots \quad \vdots \\
a_{k-1}x_k^{k-1} + \cdots + a_0 &\equiv y_k \pmod{p}
\end{aligned}
$$

Will this always work? Yes!

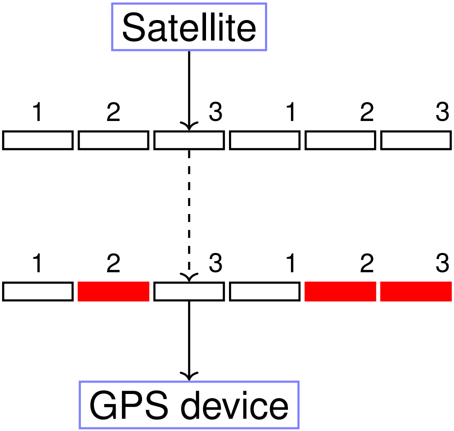
Linear algebra language: Powers of different x are linearly independent...

Also follows from polynomial properties:

Modular Arithmetic Fact: Exactly 1 polynomial of degree $\leq d$ with arithmetic modulo prime p contains $d+1$ pts.

Another Uses of Polynomials! Erasure Codes

Problem: Satellite communication is unreliable – may lose packets.
⇒ *We want to get the data even if some packets are lost (erased)*



3 packet message. So send 6!

Lose 3 out 6 packets.

Gets packets 1,1,and 3.

Exploring the Problem

Problem parameters: n packet message, channel that loses up to k packets.

"Can't get something for nothing theorem" (information theory version):
Can't send n packets of information in $< n$ packets
$\Rightarrow$ If we might lose k packets, must send $\geq n+k$ packets

We want: Any n packets should allow reconstruction of n packet message.

Where have we seen something like this.....
Any n point values allow reconstruction of degree $n-1$ polynomial.

Surely that's not just a coincidence, is it?
(*Hint: If it was, I wouldn't be standing here talking about it...*)

The Scheme

Problem: Want to send a message with n packets.

Channel: Lossy channel: loses k packets.

Question: Can you send $n+k$ packets and recover message?

Core idea: A degree $n-1$ polynomial determined by any n points!

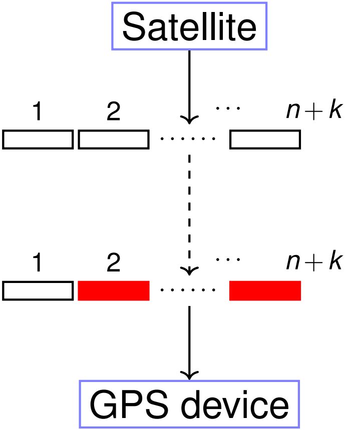
Erasure Coding Scheme: message $= m_1, m_2, \ldots, m_n$ – each b bits

1. Choose prime p a little larger than $\max(2^b, n+k)$
2. Find interpolating polynomial of $(1, m_1), (2, m_2), \ldots, (n, m_n)$
   $P(x) = a_{n-1}x^{n-1} + \cdots a_0 \pmod{p}$
3. Send $(1, P(1)), (2, P(2)), \ldots, (n+k, P(n+k))$

Any n of the $n+k$ packets gives polynomial
With polynomial, compute $P(1), P(2), \ldots, P(n)$ – the message!

Alternative: Message packets are coefficients – efficient, but less symmetric

Erasure Codes – Summary



n packet message. So send $n+k$!

Lose k packets.

Any n packets is enough!

n packet message received

Must send $n+k$ packets $\Rightarrow$ Optimal!

Transmission Efficiency

How large a p do we need? Same basic issue as in secret sharing.

Using prime p – can encode p values, so need $p \geq 2^b$ (prime so $> 2^b$)
- Can choose a prime between 2^b and 2^{b+1}
- Larger than needed, but "excess" is 1 bit per packet
- Also need to label packets, so you know which make it through

Math Magic: There are Galois Fields $GF(2^b)$ that "fit exactly"

Also need enough points for evaluation at different x (so $> n+k$)
$\Rightarrow$ Prime $p > \max(2^b, n_k)$

Information content comparison:
- Secret Sharing: each share is size of whole secret
- Erasure Coding: Each packet has size $1/n$ of the whole message

Computation time:
- Sender: Interpolation, evaluation
- Receiver: Interpolation, evaluation

No worse than $O(n^2)$ field operations (and better algorithms!)

Erasure Code: Example

Want to send 3-packet message $\langle 1, 4, 4 \rangle$

Need a polynomial through $P(1) = 1$, $P(2) = 4$, $P(3) = 4$

Interpolation... How?
- Lagrange Interpolation
- Linear System

Parameters:
- Small messages (fit in $GF(5)$)
- $n = 3$ (length of message)
- $k = 3$ (possible packets lost)

Working over $GF(p)$ — need p big enough for packets, and $p \geq n + k$

What should we use? $GF(7)$ works!

Example: Sender's Computation

Need a polynomial through $P(1) = 1$, $P(2) = 4$, $P(3) = 4$

Linear equations:

$$\begin{aligned} P(1) = a_2 + a_1 + a_0 &\equiv 1 \pmod 7 \\ P(2) = 4a_2 + 2a_1 + a_0 &\equiv 4 \pmod 7 \\ P(3) = 2a_2 + 3a_1 + a_0 &\equiv 4 \pmod 7 \end{aligned}$$

$6a_1 + 3a_0 = 2 \pmod 7$, $5a_1 + 4a_0 = 0 \pmod 7$

$a_1 = 2a_0$. $a_0 = 2 \pmod 7$ $a_1 = 4 \pmod 7$ $a_2 = 2 \pmod 7$

$P(x) = 2x^2 + 4x + 2$

$P(1) = 1$, $P(2) = 4$, and $P(3) = 4$ and $P(4) = 1$, $P(5) = 2$, and $P(6) = 0$

Send packets: $(1,1),(2,4),(3,4),(4,1),(5,2),(6,0)$

Example: Receiver's Computation

Sender sends: $(1,1),(2,4),(3,4),(4,1),(5,2),(6,0)$

Packets 3, 4, and 5 lost – receiver gets: (1,1), (2,4), (6,0)
Reconstruct?

Lagrange or linear equations:

$$\begin{aligned} P(1) &= \ \ a_2 + \ \ a_1 + a_0 &\equiv 1 \pmod 7 \\ P(2) &= 4a_2 + 2a_1 + a_0 &\equiv 4 \pmod 7 \\ P(6) &= \ \ a_2 + 6a_1 + a_0 &\equiv 0 \pmod 7 \end{aligned}$$

Solving linear equations (the magic happens...): $a_2 = 2$, $a_1 = 4$, and $a_0 = 2$
$P(x) = 2x^2 + 4x + 2$

Message? Evaluate! $P(1) = 1$, $P(2) = 4$, $P(3) = 4$
$\Rightarrow$ Message is $\langle 1,4,4 \rangle$

A Harder Problem...

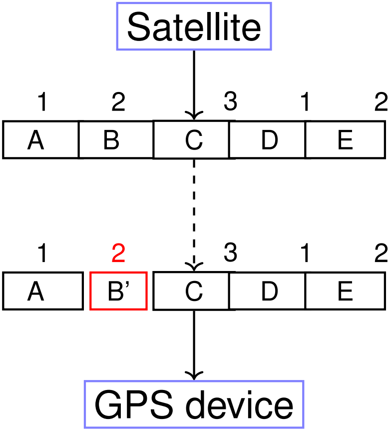
Erasure Codes:

Might completely lose packets
We know when they're missing
... and which ones are missing

Error Correction:

Noisy Channel: corrupts k packets (rather than loss)
... and no indication which ones are corrupted!

Error Correction



3 packet message. Send 5.

Corrupts 1 packets.

The Scheme

Problem: Communicate n packets $m_1, \ldots, m_n$
... on noisy channel that corrupts $\leq k$ packets

Reed-Solomon Code:

1. Make a degree $n-1$ polynomial $P(x)$ that encodes message
   - $P(1) = m_1, \ldots, P(n) = m_n$
   - Comment: could encode with packets as coefficients
2. Send $P(1), \ldots, P(n+2k)$

After noisy channel: Receive values $r_1, r_2, \ldots, r_{n+2k}$

Properties:

(1) $P(i) = r_i$ for at least $n+k$ points

(2) $P(x)$ is the unique degree $n-1$ polynomial
that contains $\geq n+k$ received points

Properties: Proof

$P(x)$: degree $n-1$ polynomial
Send $P(1), \ldots, P(n+2k)$
Receive $r_1, \ldots, r_{n+2k}$
At most k i's where $P(i) \neq r_i$.

Properties:

(1) $P(i) = r_i$ for at least $n+k$ points i,
(2) $P(x)$ is unique degree $n-1$ polynomial
that contains $\geq n+k$ of the received points.

Proof: (1) Easy – only k corruptions.
(2) Is $P(x)$ only solution?
Let $Q(x)$ be a *different* solution (deg $n-1$ contains (*any*!) $n+k$ points)

$\mathscr{Q} = \{i : Q(i) = r_i\}$ $\quad |\mathscr{Q}| \geq n+k$ $\quad |\bar{\mathscr{Q}}| \leq k$
$\mathscr{P} = \{i : P(i) = r_i\}$ $\quad |\mathscr{P}| \geq n+k$ $\quad |\bar{\mathscr{P}}| \leq k$

$|\bar{\mathscr{Q}} \cup \bar{\mathscr{P}}| \leq 2k \implies |\mathscr{Q} \cap \mathscr{P}| \geq n$
$\implies P(i) = r_i = Q(i)$ on $\mathscr{Q} \cap \mathscr{P}$ ($\geq n$ values)
$\implies Q(i) = P(i)$ at n points and degree $\leq n-1 \implies Q(x) = P(x)$ $\square$

Example: Reed-Solomon

Message: $\langle 3, 0, 6 \rangle$

Reed-Solomon Code:

Interpolation gives $P(x) = x^2 + x + 1 \pmod 7$

$P(1) = 3, P(2) = 0, P(3) = 6 \pmod 7$

Send: $P(1) = 3, P(2) = 0, P(3) = 6,$ and $P(4) = 0, P(5) = 3$

Receiver gets: $r_1 = 3, r_2 = 1, r_3 = 6, r_4 = 0, r_5 = 3$

... *2nd packet corrupted* (no indication for receiver though!)

But $n + k = 3 + 1 = 4$ points are good ($P(i) = r_i$)

Solving – The Slow Way

Brute Force!

For each subset of $n+k$ points:
  Fit degree $n-1$ polynomial, $Q(x)$, to n of them
  Check if consistent with $n+k$ of the total points
  If yes, output $Q(x)$

For a subset of $n+k$ "good points" ($r_i = P(i)$):
  Good points, so reconstructs $P(x)$ — verifies with k other good points
  All good!

For any subset of $n+k$ points:
  unique degree $n-1$ polynomial $Q(x)$ that fits $\geq n$ of them
    ... and where $Q(x)$ is consistent with $n+k$ points
      $\implies P(x) = Q(x)$.

Reconstructs $P(x)$ and only $P(x)$!!

Example

Send: $P(1) = 3, P(2) = 0, P(3) = 6, P(4) = 0, P(5) = 3$

Receiver gets: $r_1 = 3, r_2 = 1, r_3 = 6, r_4 = 0, r_5 = 3$

Goal: Find $P(x) = p_2x^2 + p_1x + p_0$ that contains $n + k = 3 + 1 = 4$ points.

All equations...

$$\begin{aligned}
p_2 + \ \ p_1 + p_0 &\equiv 3 \pmod 7 \\
4p_2 + 2p_1 + p_0 &\equiv 1 \pmod 7 \\
2p_2 + 3p_1 + p_0 &\equiv 6 \pmod 7 \\
2p_2 + 4p_1 + p_0 &\equiv 0 \pmod 7 \\
4p_2 + 5p_1 + p_0 &\equiv 3 \pmod 7
\end{aligned}$$

Assume point 1 is wrong and solve... no consistent solution!

Assume point 2 is wrong and solve... consistent solution!

With one error, only $n + 2$ error locations – for general k (location *sets*)?

The Problem For General k

$P(x) = p_{n-1}x^{n-1} + \cdots p_0$ and receive $r_1, r_2, \ldots, r_{n+2k}$

$$\begin{aligned}
p_{n-1} + \cdots p_0 &\equiv r_1 \pmod{p} \\
p_{n-1}2^{n-1} + \cdots p_0 &\equiv r_2 \pmod{p} \\
&\cdot \\
p_{n-1}i^{n-1} + \cdots p_0 &\equiv r_i \pmod{p} \\
&\cdot \\
p_{n-1}(m)^{n-1} + \cdots p_0 &\equiv r_m \pmod{p}
\end{aligned}$$

Error!! ... Where??? ... Brute Force!

Could be anywhere!!! ... so try everywhere

How many? $\binom{n+2k}{k}$ possibilities for k locations

Something like $(n/k)^k$ ... exponential in k

Can we find where the bad packets are efficiently?!?!?!

Isolating The Bad Packets

$$
\begin{aligned}
E(1)(p_{n-1} + \cdots p_0) &\equiv r_1 E(1) \pmod{p} \\
E(2)(p_{n-1}2^{n-1} + \cdots p_0) &\equiv r_2 E(2) \pmod{p} \\
&\vdots \\
E(m)(p_{n-1}(m)^{n-1} + \cdots p_0) &\equiv r_{n+2k} E(m) \pmod{p}
\end{aligned}
$$

Idea: Multiply equation i by 0 if and only if $P(i) \neq r_i$.
Blots out error locations – makes them irrelevant!
All equations satisfied!!!!!

But which equations should we multiply by 0?

We will use a polynomial!!! That we don't know. But can find!

Errors at points $e_1, \ldots, e_k$ (in diagram above, $e_1 = 2$)

Error-locator polynomial: $E(x) = (x - e_1)(x - e_2)\ldots(x - e_k)$

$E(x) = 0$ if and only if $x = e_j$ for some j

Multiply equations by $E(x)$ (above $E(x) = (x - 2)$)

All equations satisfied!!

Example

Receiver gets: $r_1 = 3, r_2 = 1, r_3 = 6, r_4 = 0, r_5 = 3$

Find $P(x) = p_2x^2 + p_1x + p_0$ that contains $n + k = 3 + 1 = 4$ of the points.

Set up linear equations...

$$
\begin{aligned}
(1 + b_0)(p_2 + p_1 + p_0) &\equiv (3)(1 + b_0) \pmod 7 \\
(2 + b_0)(4p_2 + 2p_1 + p_0) &\equiv (1)(2 + b_0) \pmod 7 \\
(3 + b_0)(2p_2 + 3p_1 + p_0) &\equiv (6)(3 + b_0) \pmod 7 \\
(4 + b_0)(2p_2 + 4p_1 + p_0) &\equiv (0)(4 + b_0) \pmod 7 \\
(5 + b_0)(4p_2 + 5p_1 + p_0) &\equiv (3)(5 + b_0) \pmod 7
\end{aligned}
$$

Error-locator polynomial: $(x - 2)$

Multiply equation i by $(i - 2)$. All equations satisfied!

But don't know the error-locator polynomial!

Do know form: $(x - e)$ or $x + b_0$

In general: $(x - e_1)(x - e_2)\ldots(x - e_k) \longrightarrow x^k + b_{k-1}x^{k-1} + \cdots b_0$

4 unknowns (p_0, p_1, p_2 and b_0), but nonlinear equations.

Nonlinear to Linear

$$E(1)(p_{n-1} + \cdots p_0) \equiv r_1 E(1) \pmod{p}$$
$$\vdots$$
$$E(i)(p_{n-1} i^{n-1} + \cdots p_0) \equiv r_i E(i) \pmod{p}$$
$$\vdots$$
$$E(m)(p_{n-1}(n+2k)^{n-1} + \cdots p_0) \equiv r_m E(m) \pmod{p}$$

$m = n+2k$ satisfied equations, $n+k$ unknowns – but nonlinear!

Let $Q(x) = E(x)P(x) = a_{n+k-1}x^{n+k-1} + \cdots a_0$

Equations:

$$Q(i) = r_i E(i)$$

... and linear in a_i and coefficients of $E(x)$!

But now more unknowns... how many?

Unknowns in $Q(x)$ and $E(x)$

$E(x)$ has degree k:

$$E(x) = x^k + b_{k-1}x^{k-1} \cdots b_0$$

$\Longrightarrow$ Leading coefficient is 1 – remaining k coefficients are unknowns

$Q(x) = P(x)E(x)$ has degree $n+k-1$:

$$Q(x) = a_{n+k-1}x^{n+k-1} + a_{n+k-2}x^{n+k-2} + \cdots a_0$$

$\Longrightarrow$ $n+k$ coefficients are unknowns

Total number of unknown coefficients: $n+2k$

Solving for $Q(x)$ and $E(x)$ ... and $P(x)$

Let $m = n+2k$ be number of points. For all points $i \in \{1,2,\ldots,m\}$,

$$Q(i) = P(i)E(i) \equiv r_i E(i) \pmod{p}$$

Gives $n+2k$ linear equations:

$$\begin{array}{rcl}
\underline{\text{From } Q(x)} & & \underline{\text{From } r_i E(x)} \\
a_{n+k-1} + \ldots a_0 & \equiv & r_1(1 + b_{k-1} + \cdots + b_0) \pmod{p} \\
a_{n+k-1}(2)^{n+k-1} + \ldots a_0 & \equiv & r_2((2)^k + b_{k-1}(2)^{k-1} + \cdots + b_0) \pmod{p} \\
 & \vdots & \\
a_{n+k-1}(m)^{n+k-1} + \ldots a_0 & \equiv & r_m((m)^k + b_{k-1}(m)^{k-1} + \cdots + b_0) \pmod{p}
\end{array}$$

... and $n+2k$ unknown coefficients of $Q(x)$ and $E(x)$!

Solve for coefficients of $Q(x)$ and $E(x)$.

Find $P(x) = Q(x)/E(x)$.

How cool is that?!?!?!

Example

Receiver gets: $r_1 = 3, r_2 = 1, r_3 = 6, r_4 = 0, r_5 = 3$

$Q(x) = E(x)P(x) = a_3x^3 + a_2x^2 + a_1x + a_0$

$E(x) = x + b_0$

$Q(i) \equiv r_iE(i) \pmod 7$

$$\begin{aligned}
a_3 + a_2 + a_1 + a_0 &\equiv 3(1 + b_0) \pmod 7 \\
a_3 + 4a_2 + 2a_1 + a_0 &\equiv 1(2 + b_0) \pmod 7 \\
6a_3 + 2a_2 + 3a_1 + a_0 &\equiv 6(3 + b_0) \pmod 7 \\
a_3 + 2a_2 + 4a_1 + a_0 &\equiv 0(4 + b_0) \pmod 7 \\
6a_3 + 4a_2 + 5a_1 + a_0 &\equiv 3(5 + b_0) \pmod 7
\end{aligned}$$

$a_3 = 1$, $a_2 = 6$, $a_1 = 6$, $a_0 = 5$, and $b_0 = -2$

$Q(x) = x^3 + 6x^2 + 6x + 5$

$E(x) = x - 2$ $\longleftarrow$ Tells us error is at $i = 2$ How cool is that?!?!?!

Example: Finishing Up

$Q(x) = x^3 + 6x^2 + 6x + 5$ and $E(x) = x - 2$

```
                x^2 +    x + 1
          ------------------------
x - 2 )   x^3  + 6 x^2 + 6 x + 5
          x^3  - 2 x^2
          ----------
                 1 x^2 + 6 x + 5
                 1 x^2 - 2 x
                 ---------------
                           x + 5
                           x - 2
                           -----
                               0
```

$P(x) = x^2 + x + 1 \pmod 7 \quad \Longrightarrow \quad$ Message is $P(1) = 3, P(2) = 0, P(3) = 6$

Error Correction: Berlekamp-Welsh

This efficient decoding algorithm is the Berlekamp-Welch algorithm

After inventors Edwyn Berlekamp and Lloyd Welch

Berkeley Connection: Berlekamp was professor at Berkeley 1971–2002

Review...

Message: $m_1, m_2, \ldots, m_n$

Sender:

Make degree $n-1$ polynomial $P(x)$ where $P(i) = m_i$

Send $n+2k$ values: $P(1), \ldots, P(n+2k)$

Receiver:

Receive $r_1, r_2, \ldots, r_{n+2k}$

Solve $n+2k$ equations, $Q(i) = r_i E(i)$ to find $Q(x) = E(x)P(x)$

Compute $P(x) = Q(x)/E(x)$

Compute $P(1), \ldots, P(n)$

About the Computed Solution

Is there one and only one $P(x)$ from the Berlekamp-Welsh algorithm?

Existence (is there one?): There is a $P(x)$ and $E(x)$ that satisfy equations.

Unique solution for $P(x)$

Uniqueness (and only one): Any solution $Q'(x)$ and $E'(x)$ have

$$\frac{Q'(x)}{E'(x)} = \frac{Q(x)}{E(x)} = P(x). \tag{1}$$

Proof: We claim (proof on next slide!)

$$Q'(x)E(x) = Q(x)E'(x) \quad \text{on } n+2k \text{ values of } x \tag{2}$$

Equation (2) implies (1). Not as easy as it seems – subtle issue to handle:

$Q'(x)E(x)$ and $Q(x)E'(x)$ are degree $n+2k-1$ and agree on $n+2k$ points

$E(x)$ and $E'(x)$ have at most k roots each (recall: roots are error locations)

So n places where neither is zero: can cross divide at n points.

$\implies \frac{Q'(x)}{E'(x)} = \frac{Q(x)}{E(x)}$ equal on n points (*Look Ma! No division by zero!*)

Both degree $\leq n-1 \implies$ Same polynomial! □

What About That Claim?

Claim: $Q'(x)E(x) = Q(x)E'(x)$ on $n+2k$ values of x.

Proof: Construction implies that for $i \in \{1,2,\ldots,n+2k\}$,

$$Q(i) = r_i E(i)$$

$$Q'(i) = r_i E'(i)$$

If $E(i) = 0$, then $Q(i) = 0$. ... and if $E'(i) = 0$, then $Q'(i) = 0$.

$\Longrightarrow$ $Q(i)E'(i) = Q'(i)E(i)$ holds when either $E(i)$ or $E'(i)$ is zero.

When $E'(i)$ and $E(i)$ are not zero (*don't divide by zero!*)

$$\frac{Q'(i)}{E'(i)} = \frac{Q(i)}{E(i)} = r_i.$$

Cross multiplying gives

$$Q'(i)E(i) = Q(i)E'(i) = r_i,$$

for these points.

So holds when $E(i)$ is zero, $E'(i)$ is zero, or neither is zero. □

Yaay!!

Berlekamp-Welsh algorithm decodes correctly when $\leq k$ errors!

Concept Check 1

Context:

You want to send a message of length 4
You construct $P(x)$ and send $P(1), P(2), \ldots, P(8)$
Receiver gets $r_1, r_2, \ldots, r_8$
Packets 1 and 4 are corrupted

Which of the following is not true?

(A) $r_1 \neq P(1)$

(B) The degree of $P(x)E(x)$ is 5

(C) The degree of $E(x)$ is 2

(D) The number of coefficients of $P(x)$ is 4

(E) The number of coefficients of $Q(x)$ is 5

Answer: (E) is false (degree 5; 6 coefficients)

Concept Check 2

Context:

You want to send a message of length 4
You construct $P(x)$ and send $P(1), P(2), \ldots, P(8)$
Receiver gets $r_1, r_2, \ldots, r_8$

Packets 1 and 4 are corrupted

Which of the following are true?

(A) $E(x) = (x-1)(x-4)$

(B) The number of coefficients in $E(x)$ is 2

(C) The number of unknown coefficients in $E(x)$ is 2

(D) $E(x) = (x-1)(x-2)$

(E) $r_4 \neq P(4)$

Answer: (A), (C), (E)

Summary

Erasure codes: Communicate n packets with k erasures.

How many packets to send? $n+k$

How to encode? With polynomial $P(x)$.

... of degree? $n-1$

Recover? Reconstruct $P(x)$ with any n points!

Error Correcting Codes (ECC): Communicate n packets with k errors.

How many packets to send? $n+2k$

How to encode? With polynomial $P(x)$.

... of degree? $n-1$

Recover?

Reconstruct error polynomial, $E(x)$, and $P(x)$! Nonlinear equations.

Reconstruct $E(x)$ and $Q(x) = E(x)P(x)$. Linear Equations.

Polynomial division! $P(x) = Q(x)/E(x)$!

Reed-Solomon codes. Welsh-Berlekamp Decoding. Optimality. Perfection!