Countability

CS70: Discrete Mathematics and Probability Theory

UC Berkeley – Summer 2025

Lecture 14

Ref: Note 11

Today

Sizes of sets

- Comparing sizes of two sets – bijections
- Infinite sets too!

How large are the common sets?

- Natural Numbers?
- Integers?
- Rationals?
- Reals?

Other infinite sets?

- Set of finite-length binary strings?
- Set of subsets of natural numbers?

Along the way:

- A new proof technique: diagonalization

Initial Poll: Sizes of Infinite Sets

Question: Which of the following are true?

(A) There are more real numbers than natural numbers. True!

(B) There are more integers than natural numbers. False!

(C) There are more rational numbers than natural numbers. False!

(D) There are more pairs of natural numbers than natural numbers. False!

Why? ... and how do these even make sense?

Comparing Apples to Oranges

Back to Kindergarten – counting sets of fruit!

Apples = { Braeburn, McIntosh, Gala, Fuji }

Oranges = { Valencia, Mandarin, Blood, Navel }

Can count each set: 4 elements each

Same size!

From Lecture 12:

Counting Rule: If there is a bijection between two sets they have the same size!

... so:

Bijection between sets, so the same size

(*maybe you didn't learn bijections in kindergarten...*)

Can we do this with *infinite* sets?

Counting

How to count?

Counting: 0, 1, 2, 3, ...

We're counting stuff... what if we don't have any stuff?
    So we need zero too
    The natural numbers: $\mathbb{N}$

Definition: A set S is **countable** if there is a bijection between S and some subset of $\mathbb{N}$.

If the subset of $\mathbb{N}$ is finite, S has finite **cardinality**.

If the subset of $\mathbb{N}$ is infinite, S is **countably infinite**.

Infinity Plus One

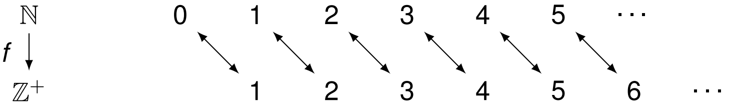
Back to grade school....

Bart: I've got 100 cookies
Lisa: Oh yeah, I've got 200 cookies!
Bart: I've got infinity cookies!!
Lisa: Oh yeah?!? I've got infinity plus one!!!! Game over!

A little more math-y:

$\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ Size? Infinity?
$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ Everything in $\mathbb{Z}^+$ – and zero. Infinity plus one?

Bijection $f : \mathbb{N} \to \mathbb{Z}^+$
$f(x) = x + 1$



Same size! (?!?!)

Subsets of Countable Sets

What about subsets of a countable set?

$\mathbb{N}$:	0	1	2	3	4	5	6	7	8	$\cdots$
	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	
S:	s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	$\cdots$
$T \subseteq S$:	s_0			s_3	s_4		s_6		s_8	$\cdots$
	$\updownarrow$			$\updownarrow$	$\updownarrow$		$\updownarrow$		$\updownarrow$	
Position:	0			1	2		3		4	$\cdots$

Theorem: If S and T are infinite, with $T \subseteq S$ and S is countably infinite, then T is countably infinite.

Note:

We don't need to know the "position" of each element in T
The position doesn't need to be *computable*
It's enough that it *exists*

Countability of $\mathbb{Z}^+$: We have $\mathbb{Z}^+ \subseteq \mathbb{N}$, so $\mathbb{Z}^+$ is countable.

Integers

$\mathbb{Z}$ has both positive and negative values – must be bigger than $\mathbb{N}$, right?

No! Interleaving bijection:

$\mathbb{N}$:	0	1	2	3	4	5	6	7	8	$\cdots$
	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	
$\mathbb{Z}$:	0	-1	1	-2	2	-3	3	-4	4	$\cdots$

Theorem: $\mathbb{Z}$ is countable.

Enumeration Example: Finite Length Binary Strings

Notation:

$B = \{0, 1\}^*$ is the set of all (finite length) binary strings

ε is the empty string

Enumeration: All length 0, then all length 1, then all length 2, ...

0: ε
1: `0`
2: `1`
3: `00`
4: `01`
5: `10`
6: `11`
7: `000`
8: `001`
9: `010`
...

Are all strings on the list?

Yes!

String $b \in \{0, 1\}^*$

Finitely many *shorter* strings

Finitely many *of the same length*

$\Rightarrow$ Finite position on list

Can even calculate position:

Take strings b, prepend a `1`

Treat as a binary number and subtract 1

Try some!

Theorem: The set of all finite length binary strings is countable.

Pairs of Numbers

Consider $S = \{1,2,3\}$

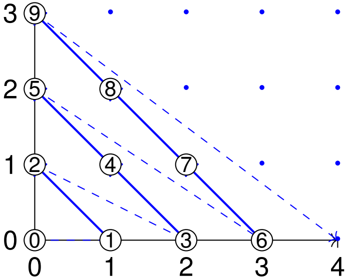
- $S \times S$ is the set of *pairs* from S
- $S \times S = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\}$
- Called the "Cartesian Product" (of S and S)
- Size? $|S \times S| = |S|^2$

What if S is infinite?

What about $\mathbb{N} \times \mathbb{N}$?

- Pairs of natural numbers
- (1,1), (4,2), (4,8), (16326324,62346124), ...
- Size? Infinity squared?

Pairs of Natural Numbers



Enumerate list:
$(0,0),(1,0),(0,1),(2,0),(1,1),(0,2),\ldots$

"Sweep d" ($d = 0, 1, \ldots$) hits all (x, y) with $x + y = d$ ... $(d+1)$ pairs

Before diagonal d:

$$\sum_{i=0}^{d-1}(i+1) = \frac{d(d+1)}{2}$$

$$f(\ (x,y)\ ) = \frac{(x+y)(x+y+1)}{2} + y$$

Check our work?

$$f(\ (0,0)\ ) = \frac{0\cdot 1}{2} + 0 = 0\ \checkmark$$
$$f(\ (2,0)\ ) = \frac{2\cdot 3}{2} + 0 = 3\ \checkmark$$
$$f(\ (1,2)\ ) = \frac{3\cdot 4}{2} + 2 = 8\ \checkmark$$

Bijection! $\Longrightarrow$ The set of pairs of natural numbers is countable

Concept Check: Bijections on Sets of Numbers

Question: Which bijection ideas work?

(A) Integers: First all negatives, then positives No! No end to negatives...

(B) Integers: By absolute value, break ties however Yes!

(C) Pairs of naturals: by sum of values, break ties however Yes!

(D) Pairs of naturals: by value of first element No! Never "increment" 2nd...

(E) Pairs of integers: by sum of values, break ties No! Negative sums?

(F) Pairs of integers: by sum of absolute values, break ties Yes!

The Rational Numbers

Consider $S = \{(x,y) \mid x,y \in \mathbb{N} \text{ and } \gcd(x,y) = 1\}$

Each $(x,y) \in S$ corresponds to a fraction $\frac{x}{y}$ in lowest terms.

Each non-negative rational written in lowest terms as $\frac{x}{y}$ and so $(x,y) \in S$.

$\Longrightarrow$ So bijection between S and $\mathbb{Q}^+$.

We also know that $S \subseteq \mathbb{N} \times \mathbb{N}$

Subset Theorem! We just proved $\mathbb{N} \times \mathbb{N}$ is countable, so S is countable.

$\Longrightarrow \mathbb{Q}^+$ is countable

What about *all* rational numbers – not just positive ones?

Let $q_0, q_1, q_2, \ldots$ be an enumeration of $\mathbb{Q}^+$

Idea: Interleave like we did mapping $\mathbb{N}$ to $\mathbb{Z}$

$\mathbb{N}$:	0	1	2	3	4	5	6	7	8	$\cdots$
	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	
$\mathbb{Q}$:	0	$-q_0$	q_0	$-q_1$	q_1	$-q_2$	q_2	$-q_3$	q_3	$\cdots$

Theorem: $\mathbb{Q}$ is countable.

The Reals

Is the set of reals $\mathbb{R}$ countable?

Lets consider the reals $[0, 1]$.

Each real has a decimal representation

`.500000000...`	$(1/2)$	
`.333333333...`	$(1/3)$	... possibly with infinite non-zero digits
`.785398162...`	$\pi/4$	... possibly with infinite non-repeating digits
`.367879441...`	$1/e$	
`.632120558...`	$1 - 1/e$	
`.345212312...`	Some number ... possibly no pattern or meaning at all	

Countable?

Can we make a numbered list of reals in $[0, 1]$?

Diagonalization

Assume for the sake of contradiction there's a mapping $\mathbb{N} \to \mathbb{R}[0,1]$:

0: .500000000...
1: .333333333...
2: .785398162...
3: .367879441...
4: .632120558...
5: .345212312...
⋮

Construct "diagonal number" – digits from diagonal, add 2 (mod 10) to each:
.757044 ...

Can the diagonal number be in the list?
Position n? No! Digit $n+1$ differs
Subtle point: Why add *two*? avoids problems like $0.25 = 0.2499999...$.

Diagonal number is a real number
Diagonal number is not in the list
The list is a list of all real numbers

Contradiction! $\Longrightarrow$ $\mathbb{R}[0,1]$ is not countable

All Reals
Could there be less?

Recall the Subset Theorem:

Theorem: If S and T are infinite, with $T \subseteq S$ and S is countably infinite, then T is countably infinite.

Contrapositive: If T is not countable, then S is not countable.
So $\mathbb{R}[0, 1]$ is uncountable $\implies$ $\mathbb{R}$ is uncountable.

All Reals

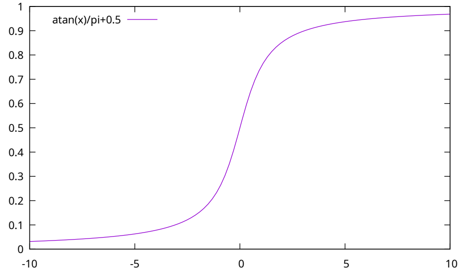
Could there be more?

Showed $\mathbb{R}[0,1]$ is uncountable – what about all of $\mathbb{R}$?

Is the set of all reals even larger?

No. "Almost" bijection, mapping open $\mathbb{R}$ to open interval $(0,1)$:

$$f(x) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}(x)$$



Complete bijection with closed interval $[0,1]$ is harder...

... but all we care about is cardinality, so 2 points don't change that.

Diagonalization

General outline of a diagonalization proof:

1. Assume that a set S can be enumerated
2. Consider an arbitrary list of all the elements of S
3. Use the diagonal from the list to construct a new element t
4. Show that t is different from all elements in the list
   $\Longrightarrow$ t is not in the list
5. Show that t is in S
6. Contradiction

Power Sets

$\mathscr{P}(S)$ is the **power set** of S – the set of all subsets of S.

Example:

$S = \{1,2,3\}$

$\mathscr{P}(S) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

Theorem: If S is finite, then $|\mathscr{P}(S)| = 2^{|S|}$.

Proof: Count them!

First element? Choose to include or not. 2 choices.
Second element? Choose to include or not. 2 choices.
...
$|S|^{\text{th}}$ element? Choose to include or not. 2 choices.
First rule of counting: $2 \times 2 \times \cdots \times 2 = 2^{|S|}$ □

What if S is infinite? How big is $\mathscr{P}(S)$?

Another Diagonalization Proof

Theorem: $\mathscr{P}(\mathbb{N})$ is not countable.

Proof: Subset representation: Infinite binary strings

Subset $S \subseteq \mathbb{N}$: Bit i is 1 if and only if $(i-1) \in S$

$\emptyset = 0000000000000...$
$\{1,4,6\} = 0100101000000...$
$\text{Evens} = 1010101010101 0...$
$\text{Primes} = 0011010100010 1...$

Assume countable, so enumeration:

```
0:  111110100111...
1:  100000110101...
2:  111011111010...
3:  000111001000...
4:  011100111100...
...
```

Make "diagonal set" – flip each bit

```
01001...
```

In listing?

At position n:

Bit $(n+1)$ is flipped

... so can't be at position n

Contradiction!

Concept Check: Review

True or false?

(A) $\mathbb{Z}$ is larger than $\mathbb{N}$ because it has negatives too False!

(B) $\mathbb{Z}$ is countable because of interleaving bijection True!

(C) $\mathbb{Q}$ is uncountable because infinitely many between 0 and 1 False!

(D) Reals in list: “diagonal number” not on list – Contradiction! True!

(E) Powerset in list: “diagonal set” not in list True!

Weirdness

Is anything bigger than $\mathscr{P}(\mathbb{N})$?

- What about $\mathscr{P}(\mathscr{P}(\mathbb{N}))$?

Talking about "levels of infinity" – use "aleph ($\aleph$) numbers"

- Cardinality of $\mathbb{N}$ is $\aleph_0$
- Cardinality of *any countable set* is $\aleph_0$
- Cardinality of $\mathscr{P}(\mathbb{N})$ is $2^{\aleph_0}$
- Cardinality of $\mathbb{R}$ is $2^{\aleph_0}$

$$\aleph_0 \quad \underbrace{\ldots\ldots\ldots}_{\text{Anything Here?}} \quad 2^{\aleph_0}$$

No ... and yes ...

- It's complicated ("Continuum Hypothesis")

Work of logician Kurt Gödel – old-ish book: *Gödel, Escher, Bach*

A Peak Ahead

How many predicates $P : \mathbb{N} \rightarrow \{\text{True, False}\}$?

Ex: Is x even? Is x a perfect square? Is x prime? Does Collatz hold for x?

Same as number of subsets of $\mathbb{N}$ (subset elements $\leftrightarrow$ True values)

$\Longrightarrow$ Set of predicates is uncountable

How many programs?

A program is a finite length binary string

Set of finite length binary strings is countable

$\Longrightarrow$ Set of programs is countable

So: More (*many more*) predicates than programs

$\Longrightarrow$ Programs can't compute all predicates

$\Longrightarrow$ There are uncomputable functions

Lecture 14 Summary

Sizes of sets

- Comparing sizes of two sets – bijections
- Infinite sets too!

Countably infinite sets

- Counting numbers ($\mathbb{N}$) are countable (surprise!)
- Integers are countable
- Rationals are countable
- The set of all finite-length binary strings is countable

Uncountable sets

- The set of reals is uncountable
- Diagonalization as a proof technique
- The set of subsets of $\mathbb{N}$ is uncountable

Bottom line: Infinity is weird. And cool.